Probability II: B. Math (Hons.) I
Academic Year 2022-23, Second Semester
Final Exam
Total Marks = 50 Duration: 10:00 am - 1:00 pm

- **Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. A continuous random vector (X, Y) has a joint probability density function given by
$$f_{X,Y}(x, y) = e^{-(x-y)} \quad \text{if } 0 < y < 1 \text{ and } x > y.$$
   (a) (1 mark) Are X and Y independent? Please justify your answer.
   (b) (4+4 = 8 marks) Calculate the conditional probability density functions of X given Y and Y given X.

2. (10 marks) Let N be the number of empty poles when r distinguishable flags are displayed at random on n distinguishable poles (here $r, n \in \mathbb{N}$). Assuming that each pole has unlimited capacity, compute the variance of N.

3. (5+(3+3) = 11 marks) If $X \sim Bin(m, p)$ and $Y \sim Bin(n, p)$ are independent, then find the conditional distribution of X given $X + Y$. In this case, compute $E(X|X + Y)$ and verify (by direct computation) that $E\big[E(X|X + Y)\big] = E(X)$.

4. (10 marks) Suppose $(X_1, X_2, X_3, X_4, X_5) \sim Dir(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; \alpha_6)$ (the notation is as used in the class), where $\alpha_1, \alpha_2, \ldots, \alpha_6$ are strictly positive parameters. Compute, with full justification, a joint probability density function of $(X_5 + X_1, X_3, X_2)$.

5. Suppose X and Y are jointly normal (i.e., bivariate normal) with $E(X) = E(Y) = 0$, $Var(X) = Var(Y) = 1$ and $Corr(X, Y) = \rho \in (-1, 1)$.
   (a) (5 marks) Show that $(X, Y)^T \stackrel{d}{=} \big(Z,\ \rho Z + \sqrt{1-\rho^2}\, W\big)^T$, where $Z, W \stackrel{iid}{\sim} N(0, 1)$.
   (b) (5 marks) Using (a) or otherwise, compute, with full justification, $E\big(X^2Y^2\big)$.